Composing Stepsize Schedules for Minimizing Smooth Convex Functions

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Joint work with:
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- Notation: $f_i = f(x_i)$ and $g_i = \nabla f(x_i)$

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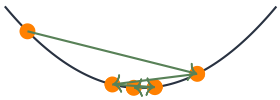

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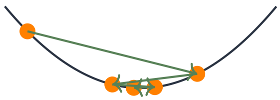

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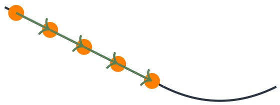

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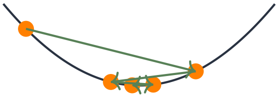

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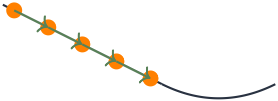

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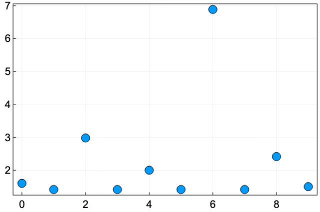

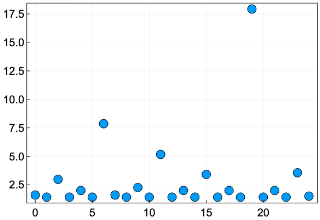

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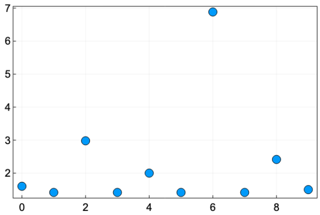
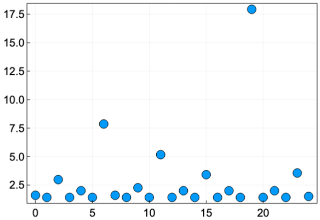
  Identification of conjectured optimal $\mathbf{h}$ for all N

$$\sup_{(f, x_0)} \frac{f_N - f_\star}{\frac{1}{2}\|x_0 - x_\star\|^2} = O\left(\frac{1}{N^{1.2716}}\right)$$

where $1.2716 = \log_2(1 + \sqrt{2})$





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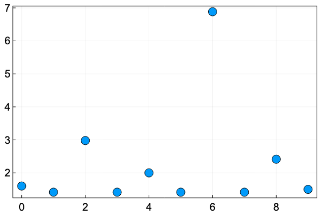
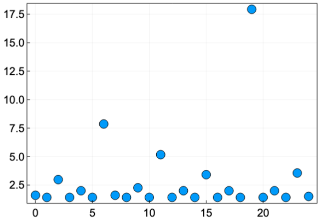
  Additional extensions





Definitions and Results

Composable Sequences

Definition

$\mathbf{h}$ is f-composable with rate η if

$$\sup_{(f,x_0)} \frac{f_N - f_\star}{\frac{1}{2}\left\|x_0 - x_\star\right\|^2} = \eta \quad \text{where } \eta = \frac{1}{1 + 2\sum_i h_i} = \prod_i (h_i - 1)^2$$

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Example: $[\,]$ is f-composable and s-composable with rate 1

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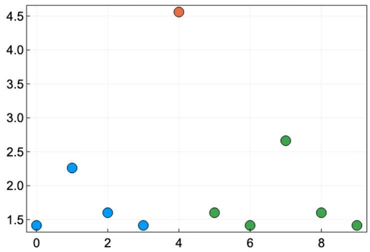

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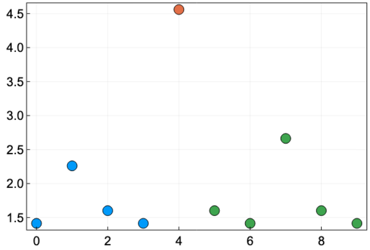

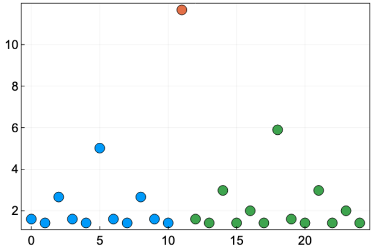

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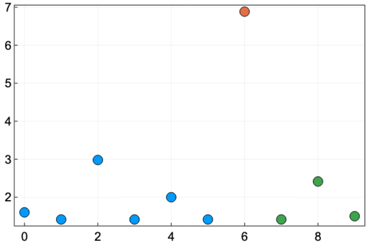

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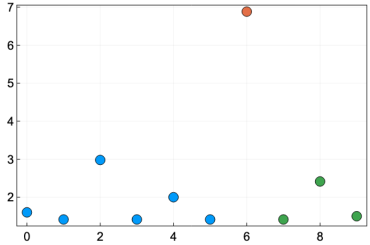

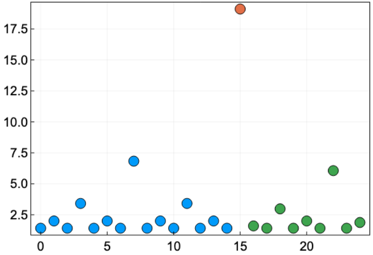

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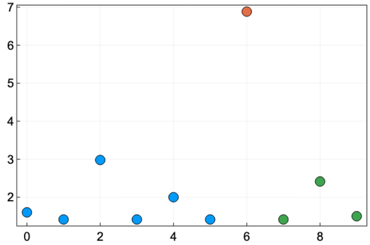
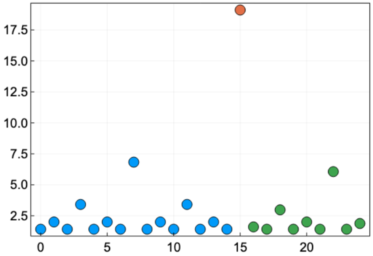
$\mathbf{h}$ is basic if it can be constructed from $[\,]$, $\bowtie$, $\triangleright$





Applications: Silver Stepsizes

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\begin{aligned}
\mathbf{h}^{(0)} &= [\,] \\
\mathbf{h}^{(1)} &= [\sqrt{2}] \\
\mathbf{h}^{(2)} &= [\sqrt{2}, 2, \sqrt{2}] \\
\mathbf{h}^{(3)} &= [\sqrt{2}, 2, \sqrt{2}, 2+\sqrt{2}, \sqrt{2}, 2, \sqrt{2}] \\
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\mathbf{h}^{(3)} &= [\sqrt{2}, 2, \sqrt{2}, 2+\sqrt{2}, \sqrt{2}, 2, \sqrt{2}] && = \mathbf{h}^{(2)} \bowtie \mathbf{h}^{(2)} \\
&\dots
\end{aligned}
$$

Applications: Silver Stepsizes

- The silver stepsize schedules of [Altschuler, Parrilo '23a,b]:

$$
\begin{aligned}
\mathbf{h}^{(0)} &= [\,] && = [\,] \\
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&\dots
\end{aligned}
$$

- The sequence $\mathbf{h}^{(k)}$ is basic s-composable with rate $\eta^{(k)} = (1+\sqrt{2})^{-k}$

$$
\sup_{(f,x_0)} \frac{f_{2^k-1} - f_\star}{\frac{1}{2}\left\|x_0 - x_\star\right\|^2} = \frac{\eta^{(k)}}{2-\eta^{(k)}} \approx \frac{1}{2(1+\sqrt{2})^k}
$$

Applications: Numerically Optimal Schedules

- Numerically optimal stepsize schedules of **[Das Gupta, Van Parys, Ryu '23]**:

$$
\begin{aligned}
\mathbf{h}^{(0)} &= [\,] \\
\mathbf{h}^{(1)} &= [3/2] \\
\mathbf{h}^{(2)} &= [1.4142, 1.8767] \\
\mathbf{h}^{(3)} &= [1.4142, 2.4142, 1.5] \\
\mathbf{h}^{(4)} &= [1.4142, 1.6012, 3.0051, 1.5] \\
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  - All $\mathbf{h}^{(n)}$ for $n = 1, \dots, 25$ are basic f-composable

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  - Most $\mathbf{h}^{(n)}$ actually $\approx 1\%$ *suboptimal* for $n = 6, \dots, 25$

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  - All $\mathbf{h}^{(n)}$ for $n = 1, \dots, 25$ are basic f-composable
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- **Conjecture**: the local minimizers of

$$
\min_{\mathbf{h}} \sup_{(f, x_0)} \frac{f_N - f_\star}{\frac{1}{2}\left\|x_0 - x_\star\right\|^2}
$$

are basic f-composable schedules

Applications: Dynamic Short Stepsizes

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- Compare with **[Teboulle, Vaisbourd '22] [Rotaru, Glineur, Patrinos '24]**

Applications: Optimal Basic Schedules

Optimal basic f-composable schedule of length N can be computed by DP

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$$\mathbf{h}_{\mathsf{f}}^{(n)} = \min_{m=0,\ldots,n-1} \mathbf{h}_{\mathsf{s}}^{(m)} \triangleright \mathbf{h}_{\mathsf{f}}^{(n-m-1)}$$

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Theorem

The optimal basic f-composable schedule of length N has guarantee:

$$\sup_{(f,x_0)} \frac{f_N - f_\star}{\frac{1}{2}\left\|x_0 - x_\star\right\|^2} \leq \frac{0.42311 + o(1)}{(N+1)^{\log_2(1+\sqrt{2})}}$$

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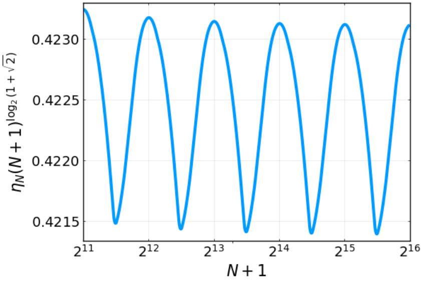

Summary and Pointers

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Thank you for listening!
Read more: `arXiv:2410.16249`
Questions?

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Appendix:
Overview of Composition Proofs

Key lemma

$\mathbf{h}$ is f-composable iff $f_N - f_\star \leq \frac{\eta}{2}\|x_0 - x_\star\|^2$ and tight for Huber and quadratic

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Lemma

If $\mathbf{h}$ is f-composable with rate η, then for any z and any $t \in [0,1]$, it holds that

$$f_N - f_\star \leq \frac{\eta}{2}\left\|x_0 - (1-t)x_\star - tz^+\right\|^2 + t\left(f(z)^+ - f_\star\right)$$

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Proof strategy.

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Proof strategy.

- PEP certificate: $f_\star - f_N + \frac{\eta}{2}\|x_0 - x_\star\|^2 = \sum_{i,j} \lambda_{i,j} Q_{i,j} + (\succeq 0)$

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- Replace $Q_{\star,j}$ with $(1-t)Q_{\star,j} + tQ_{z,j} \geq 0$ $\square$

F-join of two sequences

- $\mathbf{a}$ is s-convergent with rate α

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$$\begin{aligned} &\boxed{x_0} \text{ is given} \\ &x_{i+1} = x_i - a_i g_i \\ &\boxed{x_M} = x_{M-1} - a_{M-1} g_{M-1} \end{aligned}$$

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$$y_0 = x_M - \mu g_M$$

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$$\boxed{x_M} = x_{M-1} - a_{M-1} g_{M-1}$$

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&x_M = x_{M-1} - a_{M-1} g_{M-1} && y_N = y_{N-1} - b_{N-1} \ell_{N-1}
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$$

- **Strategy**:

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- **Strategy**:
  - Use s-composable definition for $\mathbf{a}$

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&\boxed{x_0} \text{ is given} && \boxed{y_0 = x_M - \mu g_M} \\
&x_{i+1} = x_i - a_i g_i && y_{i+1} = y_i - b_i \ell_i \\
&\boxed{x_M} = x_{M-1} - a_{M-1} g_{M-1} && \boxed{y_N} = y_{N-1} - b_{N-1} \ell_{N-1}
\end{aligned}
$$

- **Strategy**:
  - Use s-composable definition for $\mathbf{a}$
  - Use upgraded f-composable inequality for $\mathbf{b}$ and compare with $z = x_M$

Proof sketch

- s-composability of $\mathbf{a}$:

$$f_M - f_\star \leq \frac{\alpha}{2-\alpha}\left(\frac{1}{2}\left\|x_0 - x_\star\right\|^2 - \frac{1}{2}\left\|x_M - \frac{g_M}{\alpha} - x_\star\right\|^2\right)$$

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- Key lemma applied to $\mathbf{b}$ with $z = x_M$

$$f_N - f_\star \leq \frac{\beta}{2}\|(1-t)(x_M - x_\star) - (\mu - t)g_M\|^2 - \frac{t}{2}\|g_M\|^2 + t\left(f_M - f_\star\right)$$

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Pick $t = 1 + \alpha(1-\mu)$ so that the norm terms cancel out:

$$f_N - f_\star \leq \frac{\alpha^2\beta(\mu-1)^2}{2}\|x_0 - x_\star\|^2 + (\text{expression in } f_M - f_\star, x_M - x_\star, g_M)$$

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- Ask Mathematica to optimize rate:

$$\mu = 1 + \frac{\sqrt{\alpha^2 + 8\alpha\beta} - \alpha}{4\alpha\beta}$$

$$\text{new rate} = \frac{2\alpha\beta}{\alpha + 4\beta + \sqrt{\alpha^2 + 8\alpha\beta}} = \alpha \triangleright \beta. \qquad \square$$